

DIOPHANTINE PROPERTIES OF MEASURES INVARIANT WITH RESPECT TO THE GAUSS MAP

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ABSTRACT. Motivated by the work of D. Y. Kleinbock, E. Lindenstrauss, G. A. Margulis, and B. Weiss [7, 8], we explore the Diophantine properties of probability measures invariant under the Gauss map. Specifically, we prove that every such measure which has finite Lyapunov exponent is extremal, i.e. gives zero measure to the set of very well approximable numbers. We show on the other hand that there exist examples where the Lyapunov exponent is infinite and the invariant measure is not extremal. Finally, we answer in the negative a question posed by Kleinbock, Lindenstrauss, and Weiss [7], by constructing a family of measures on \mathbb{R} which are Ahlfors regular and yet do not satisfy a 0-1 law for approximability.

1. INTRODUCTION

Definition 1.1. Let $\psi : \mathbb{N} \rightarrow (0, \infty)$ be any function. We recall that an irrational $x \in [0, 1]$ is ψ -*approximable* if there exist infinitely many $p/q \in \mathbb{Q}$ such that

$$(1.1) \quad \left| x - \frac{p}{q} \right| \leq \psi(q).$$

We recall the following facts and definitions from the classical theory of Diophantine approximation:

- Every x is ψ -approximable when $\psi(q) = q^{-2}$.
- x is *badly approximable* if there exists $\varepsilon > 0$ such that x is not ψ -approximable when $\psi(q) = \varepsilon q^{-2}$. The set of badly approximable numbers has Hausdorff dimension one but Lebesgue measure zero.
- x is *very well approximable* if there exists $c > 0$ such that x is ψ -approximable when $\psi(q) = q^{-(2+c)}$. The set of very well approximable numbers has Hausdorff dimension one but Lebesgue measure zero.
- x is a *Liouville* number if for all $c > 0$ the number x is ψ -approximable when $\psi(q) = q^{-c}$. The set of Liouville numbers has Hausdorff dimension zero.

1.1. Extremal measures. A measure¹ μ on \mathbb{R} is said to be *extremal* if the set of very well approximable numbers is null with respect to μ . In other words, μ behaves like Lebesgue measure with respect to very well approximable numbers. This definition was introduced by D. Y. Kleinbock, E. Lindenstrauss, and B. Weiss in [7], as a generalization of the notion of an extremal manifold, which was defined by V. Sprindžuk. B. Weiss [12] proved that measures which satisfy a certain decay condition, called *absolutely decaying*, are extremal.

Definition 1.2. For $\alpha > 0$, a measure μ on \mathbb{R} is said to be *absolutely α -decaying* if there exists $C > 0$ such that for all $x \in \mathbb{R}$, for all $0 < r \leq 1$ and for all $0 < \varepsilon \leq 1$ we have

$$(1.2) \quad \mu(B(x, \varepsilon r)) \leq C \varepsilon^\alpha \mu(B(x, r)).$$

It is said to be *absolutely decaying* if it is absolutely α -decaying for some $\alpha > 0$.

We recall also that for $\delta > 0$, a measure μ on \mathbb{R} is *Ahlfors δ -regular* if there exist positive constants C_1 and C_2 such that

$$C_1 r^\delta \leq \mu(B(x, r)) \leq C_2 r^\delta$$

for all x in the topological support of μ and for all $0 < r \leq 1$. Examples of Ahlfors regular measures include Lebesgue measure and the Hausdorff measure on certain fractals such as the Cantor set. Clearly, any Ahlfors δ -regular measure on \mathbb{R} is automatically absolutely δ -decaying.

Generalizations of Weiss's result to higher dimensions have been considered by Kleinbock, Lindenstrauss, and Weiss [7]. However, for the purposes of this paper we will consider only Weiss's original result and not the higher dimensional generalizations.

Let $G : [0, 1] \rightarrow [0, 1]$ be the Gauss map, i.e.

$$(1.3) \quad G(x) = \begin{cases} \frac{1}{x} - [1/x] & x > 0 \\ 0 & x = 0 \end{cases},$$

where $[x]$ is the integer part of x . A measure μ is *invariant* with respect to the Gauss map if $\mu \circ G^{-1} = \mu$. In this paper we consider the extremality of probability measures invariant with respect to the Gauss map. Specifically, we show that if an invariant measure μ has finite Lyapunov exponent, then μ is extremal.

¹In this paper all measures are assumed to be Borel and locally finite.

Definition 1.3. If μ is a probability measure on $[0, 1]$ invariant with respect to the Gauss map, then the integral

$$\chi_\mu(G) = \int \log |G'| d\mu$$

is called the *Lyapunov exponent* of the measure μ with respect to the Gauss map G .

Theorem 2.1. *If μ is a probability measure on $[0, 1] \setminus \mathbb{Q}$ invariant with respect to the Gauss map G with finite Lyapunov exponent $\chi_\mu(G)$, then μ is extremal.*

The assumption that $\chi_\mu(G) < \infty$ is a very reasonable assumption which is satisfied for a large class of dynamically defined measures; see Section 4. In particular there exist measures which satisfy this assumption but are not absolutely decaying. It is also a necessary assumption, as seen from the following:

Theorem 4.5. *There exists a measure μ invariant with respect to the Gauss map which gives full measure to the Liouville numbers. In particular, μ is not extremal.*

1.2. A question about absolutely decaying measures. In [7], Kleinbock, Lindenstrauss, and Weiss asked the following question:²

Question 1.4 (Question 10.1 of [7]). *Suppose that μ is an absolutely decaying measure on \mathbb{R} .*

- (a) *Is it true that for any decreasing function $\psi : \mathbb{N} \rightarrow (0, +\infty)$, either the set of ψ -approximable numbers or its complement has μ -measure 0?*
- (b) *Is it true that for all ψ as in (a), μ -almost every point is ψ -approximable if and only if*

$$(1.4) \quad \sum_{q=1}^{\infty} q\psi(q) = \infty?^3$$

We answer this question in the negative by constructing a family of measures on \mathbb{R} which are Ahlfors regular (and in particular absolutely decaying) and yet do not satisfy either (a) or (b).

To construct these measures, we fix a set $I \subseteq \mathbb{N}$ and let

$$J_I = \{x \in [0, 1] \setminus \mathbb{Q} : \text{the continued fraction entries of } x \text{ lie in } I\}.$$

²Actually, Kleinbock, Lindenstrauss, and Weiss's question was about friendly measures on \mathbb{R}^d . When restricted to one dimension, friendly measures are the same as absolutely decaying measures (see Lemma 2.2 in [7]).

³In this formula we have replaced $\psi(q)$ by $q\psi(q)$ due to a difference in the definition of ψ -approximability.

Theorem 6.1. *Fix an infinite set $I \subseteq \mathbb{N}$, and let h be the Hausdorff dimension of J_I . Assume that the h -dimensional Hausdorff measure \mathcal{H}^h restricted to J_I is Ahlfors h -regular. Let $\mu = \mathcal{H}^h \upharpoonright_{J_I}$, and let $\psi : \mathbb{N} \rightarrow (0, +\infty)$ be a function such that the function $q \mapsto q^2\psi(q)$ is nonincreasing. Then μ -almost every (resp. μ -almost no) point is ψ -approximable, assuming that the series*

$$(6.1) \quad \sum_{q=1}^{\infty} q^{2\alpha-1}\psi(q)^{\alpha}$$

diverges (resp. converges).

We note that the convergence case of Theorem 6.1 is a theorem of Weiss [12], which he proved for any absolutely decaying measure μ and for any function $\psi : \mathbb{N} \rightarrow (0, +\infty)$.

Note that when $I = \mathbb{N}$, then μ is Lebesgue measure on $[0, 1]$, and Theorem 6.1 reduces to the classical Khinchine theorem.

It is not immediately obvious that the hypotheses of Theorem 6.1 can ever be satisfied (except for the case $I = \mathbb{N}$). Nevertheless we will demonstrate the following:

Theorem 7.1. *For every $0 < \delta \leq 1$ there exists an infinite set $I \subseteq \mathbb{N}$ such that $\text{HD}(J_I) = \delta$ and such that $\mathcal{H}^{\delta} \upharpoonright_{J_I}$ is Ahlfors δ -regular.⁴*

Combining Theorems 6.1 and 7.1 in the obvious way yields the following corollary:

Corollary 1.5. *For every $0 < \alpha \leq 1$, there exists an Ahlfors α -regular, and therefore absolutely α -decaying, measure μ such that for any function $\psi : \mathbb{N} \rightarrow (0, +\infty)$ such that the function $q \mapsto q^2\psi(q)$ is nonincreasing, then μ -almost every (resp. μ -almost no) point is ψ -approximable, assuming that the series (6.1) diverges (resp. converges).*

In the case $\alpha = 1$, the measure is simply Lebesgue measure.

Corollary 1.6. *The answer to Question 1.4 is negative (for both parts (a) and (b)).*

Proof. Fix $0 < \alpha < 1$ and let μ be the measure guaranteed by Corollary 1.5. To see that the answer to (b) is negative, we merely note the existence of a function ψ for which (1.4) converges but (6.1) diverges. For example,

$$\psi(q) = \frac{1}{q^2 \log^{1/\alpha}(q)}.$$

⁴Here and from now on $\text{HD}(S)$ denotes the Hausdorff dimension of a set S . $\mathcal{H}^{\delta}(S)$ and $\mathcal{P}^{\delta}(S)$ denote its δ -dimensional Hausdorff and packing measure, respectively.

To see that the answer to (a) is negative, let $y \in \mathbb{R}$ be chosen at random with respect to Lebesgue measure. As noted in [7] (see the paragraph immediately following Question 10.1), the measure $\nu := \mu \circ (x \mapsto x + y)^{-1}$ does satisfy (b) of Question 1.4. But then the measure $\mu + \nu$ is also Ahlfors regular, but does not satisfy (a); indeed, for the function ψ given above, μ -almost every point but ν -almost no point is ψ -approximable. \square

In the process of proving Theorems 6.1 and 7.1, we establish the following criterion for determining whether $\mathcal{H}^\alpha \upharpoonright_{J_I}$ is Ahlfors regular. This improves more complicated criteria which can be found in [9].

Theorem 5.5 (Abridged). *Fix an infinite set $I \subseteq \mathbb{N}$, and let $h = \text{HD}(J_I)$. The following are equivalent:*

- (a) $\mathcal{H}^h(J_I) > 0$ and $\mathcal{P}^h(J_I) < \infty$.
- (b1) $\mathcal{H}^h \upharpoonright_{J_I}$ is Ahlfors h -regular.
- (c1) For all $y \in I$ and $r \geq 1$

$$\#(B(y, r) \cap I) \asymp r^h.$$

Thus the Ahlfors regularity of J_I is equivalent to the “dual Ahlfors regularity” of the generating set I .

Note that it is possible for (c1) to be satisfied for some $h \neq \text{HD}(J_I)$. In such a case, the set J_I is not Ahlfors regular.

The structure of the paper is as follows: In Section 2, we will prove Theorem 2.1. In Section 3, we will recall some basic definitions and theorems from the theory of conformal iterated function systems, which are needed to prove Theorems 4.5, 5.5, 6.1, and 7.1. In Section 4, we will give some examples of measures which satisfy the hypotheses of Theorem 2.1, and we shall prove Theorem 4.5. In Section 5 we will discuss various characterizations of Ahlfors regularity and semiregularity of J_I , and we shall prove Theorem 5.5. In Section 6 we shall prove Theorem 6.1 and in Section 7 we shall prove Theorem 7.1.

The interdependence of the sections is as follows: Section 4 depends on Sections 2 and 3; Section 5 depends on Section 3; Section 6 depends on Sections 2, 3, and 5; Section 7 depends on 3 and 5.

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2. PROOF OF THEOREM 2.1

In this section we will prove the following theorem:

Theorem 2.1. *If μ is a probability measure on $[0, 1] \setminus \mathbb{Q}$ invariant with respect to the Gauss map G with finite Lyapunov exponent $\chi_\mu(G)$, then μ is extremal.*

2.1. A relation between continued fractions and Diophantine approximation. As the first step in the proof of Theorem 2.1, we establish a relation between the continued fraction expansion of an irrational $x \in [0, 1]$ with its Diophantine properties. This relation may be compared to Theorem 8.5 of [6].

Definition 2.2. For a function $\psi : \mathbb{N} \rightarrow (0, \infty)$, let us say that x is ψ -well approximable if x is $\varepsilon\psi$ -approximable for every $\varepsilon > 0$.

Remark 2.3. ψ -well approximability implies ψ -approximability but not vice-versa; for example, if x is a badly approximable number and $\psi(q) = 1/q^2$ then x is ψ -approximable but not ψ -well approximable.

Lemma 2.4. *Fix an irrational $x \in [0, 1]$ and let $[0; \omega_0, \omega_1, \dots]$ be the continued fraction expansion of x . Let $(p_n/q_n)_{n=0}^\infty$ be the convergents of x . Let $\psi : \mathbb{N} \rightarrow (0, \infty)$ be a function satisfying*

$$\psi(q) \leq \frac{1}{q^2}$$

for all q . Then x is ψ -well approximable if and only if for every $K > 0$ there exist infinitely many $n \in \mathbb{N}$ such that

$$(2.1) \quad \omega_n \geq K\phi(q_n),$$

where

$$\phi(q) := \frac{1}{q^2\psi(q)} \geq 1.$$

Proof. Fix an irrational $x \in [0, 1]$. We recall the following well-known facts (see e.g. [5] or [1]):

- (i) If p/q is a rational approximation of x such that $|x - p/q| < 1/(2q^2)$, then p/q is a convergent of x .
- (ii) For every $n \in \mathbb{N}$ we have

$$(2.2) \quad \frac{1}{q_n(q_n + q_{n+1})} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}$$

and

$$(2.3) \quad q_{n+1} = \omega_n q_n + q_{n-1}.$$

From (i), it follows that for any $0 < \varepsilon \leq 1/2$, (1.1) cannot be satisfied for any p/q which is not a convergent. Thus, we may restrict our

attention to approximations of x which are convergents. Fix $n \in \mathbb{N}$, and note that by (2.3) we have

$$q_{n+1} \asymp {}^5\omega_n q_n$$

and thus (2.2) implies

$$\left| x - \frac{p_n}{q_n} \right| \asymp \frac{1}{q_n^2 \omega_n}.$$

Thus, x is ψ -well approximable if and only if for every $\varepsilon > 0$ there exist infinitely many $n \in \mathbb{N}$ such that

$$\frac{1}{q_n^2 \omega_n} \geq \varepsilon \psi(q_n) = \frac{\varepsilon}{q_n^2 \phi(q_n)}.$$

(We are using the ε to absorb the constant coming from the asymptotic.) Rearranging and letting $K = 1/\varepsilon$ yields the lemma. \square

Definition 2.5. For $x \in [0, 1]$, let

$$\xi(x) = \lfloor 1/x \rfloor$$

be the first entry in the continued fraction expansion of x , so that $\xi(G^n(x)) = \omega_n$ for all n . Let

$$\eta = \log(1 + \xi).$$

Corollary 2.6. Fix an irrational $x \in [0, 1]$ and let $[0; \omega_0, \omega_1, \dots]$ be the continued fraction expansion of x . Then the following are equivalent:

- (i) The number x is very well approximable.
- (ii) There exists $c > 0$ such that for infinitely many $n \in \mathbb{N}$,

$$(2.4) \quad \log(1 + \omega_n) \geq c \sum_{j=0}^{n-1} \log(1 + \omega_j),$$

or equivalently,

$$(2.5) \quad \eta(G^n(x)) \geq c \sum_{j=0}^{n-1} \eta(G^j(x)).$$

Formula (2.5) will be more useful than (2.4) for our ergodic theory purposes.

⁵Here and from now on \asymp denotes a multiplicative asymptotic.

Proof. We first give some bounds for q_n in terms of the continued fraction entries $\omega_0, \dots, \omega_{n-1}$. The upper bound is easy: the recursion equation (2.3) implies that

$$q_n \leq \prod_{j=0}^{n-1} (\omega_j + 1).$$

In the other direction, we divide into cases according to whether n is even or odd. If $n = 2k$ then

$$q_n \geq \prod_{j=0}^{k-1} (\omega_{2j}\omega_{2j+1} + 1) \geq \prod_{j=0}^{n-1} \sqrt{\omega_j + 1}$$

and if $n = 2k + 1$ then

$$q_n \geq \omega_{n-1} q_{n-1} \geq \frac{1}{\sqrt{2}} \prod_{j=0}^{n-1} \sqrt{\omega_j + 1}.$$

Let $t \geq 0$. Taking logarithms, we can rewrite the above inequalities as

$$(2.6) \quad \frac{1}{2} \sum_{j=0}^{n-1} \eta(G^j(x)) - \log(\sqrt{2}) \leq \log(q_n) \leq \sum_{j=0}^{n-1} \eta(G^j(x)).$$

Now the corollary follows immediately from (2.6) together with Lemma 2.4, and the following characterization of the set VWA:

An irrational $x \in [0, 1]$ is very well approximable if and only if there exists $c > 0$ such that it is ψ -well approximable where $\psi(q) = q^{-(2+c)}$.

□

Using (2.6) and Lemma 2.4, we also deduce the following:

Corollary 2.7. *Fix an irrational $x \in [0, 1]$. Then the following are equivalent:*

- (i) x is a Liouville number.
- (ii) For all $c > 0$, there exist infinitely many $n \in \mathbb{N}$ such that (2.5) holds.

Remark 2.8. The well-known fact that an irrational $x \in [0, 1]$ is badly approximable if and only if its continued fraction entries are bounded is also a corollary of Lemma 2.4.

2.2. Proof of Theorem 2.1. We first of all note that it suffices to consider the case where μ is ergodic with respect to G , since if χ_μ is finite, then χ_ν is finite for almost all measures ν in the ergodic decomposition of μ . Since the set VWA is invariant with respect to the Gauss map, it follows that any ergodic measure must give either zero or full measure to VWA.

Let μ be an ergodic invariant measure whose Lyapunov exponent $\chi_\mu = \int \log |G'| d\mu$ is finite. Let η be as in Definition 2.5. Since $\eta(x) \asymp -2 \log(x) = \log |G'(x)|$, it follows that $\int \eta d\mu$ is also finite. On the other hand, η is a strictly positive function and so we have

$$(2.7) \quad 0 < \int \eta d\mu < \infty.$$

We claim that μ is extremal. Suppose to the contrary that μ -almost every number $x \in [0, 1]$ is very well approximable. It then follows from Corollary 2.6, (2.7), and the Birkhoff ergodic theorem that for μ -almost all such numbers x , we have that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \eta(G^j(x)) \leq \frac{1}{c_x} \limsup_{n \rightarrow \infty} \frac{1}{n} \eta(G^n(x)) = 0$$

with some $c_x > 0$ coming from Corollary 2.6. Invoking the Birkhoff ergodic theorem again, we conclude that $\int \eta d\mu \leq 0$. This contradiction finishes the proof.

3. ITERATED FUNCTION SYSTEMS AND CONFORMAL MEASURES

Our main example of a measure invariant with respect to the Gauss map will be the unique invariant probability measure absolutely continuous to a conformal measure associated with an iterated function system consisting of inverse branches of the Gauss map. In this section we recall the definitions and main theorems. All theorems in this section except for those in Subsection 3.5 were proven first in [9] and also in a more general context in [10].

3.1. IFSs and continued fractions. For each $i \in \mathbb{N}$, consider the map $g_i : [0, 1] \rightarrow [0, 1]$ defined by

$$g_i(x) = \frac{1}{i+x}.$$

The map g_i is an inverse branch of the Gauss map G . For any set $I \subseteq \mathbb{N}$, the collection of maps $\mathcal{S}_I = \{g_i\}_{i \in I}$ is a conformal iterated function system (see [9] or [10] for the definition).

Given $\omega = \omega_0\omega_1\omega_2\ldots\omega_{n-1} \in \mathbb{N}^n$, let

$$g_\omega := g_{\omega_0} \circ g_{\omega_1} \cdots \circ g_{\omega_{n-1}} : [0, 1] \rightarrow [0, 1],$$

so that

$$g_\omega(x) = \frac{1}{\omega_0 + \frac{1}{\omega_1 + \frac{1}{\ddots + \frac{1}{\omega_{n-1} + x}}}}.$$

In particular,

$$g_\omega(0) = [0; \omega_0, \omega_1, \ldots, \omega_{n-1}].$$

Let

$$J_I = \bigcap_{n \in \mathbb{N}} \bigcup_{\omega \in I^n} g_\omega([0, 1]).$$

The set J_I is called the *limit set* of the IFS \mathcal{S}_I . It coincides with the set of all irrational numbers in $[0, 1]$ whose continued fraction entries all lie in I . If I is infinite, then the set $\overline{J_I} \setminus J_I$ consists of the set of all rational numbers whose continued fraction entries all lie in I . Moreover, J_I is forward invariant under the Gauss map G , i.e.

$$G(J_I) = J_I.$$

3.2. A formula for the Hausdorff dimension of J_I . Fix $I \subseteq \mathbb{N}$. A famous formula of R. Bowen relates the Hausdorff dimension of J_I to an invariant of the IFS \mathcal{S}_I called the topological pressure. Below we describe this invariant and state Bowen's formula.

Given $t \geq 0$, the following limit exists and is called the *topological pressure* of the IFS \mathcal{S}_I at the parameter t :

$$P_I(t) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\omega \in I^n} \|g'_\omega\|_\infty^t.$$

Theorem 3.1 (Bowen's formula; Theorem 4.2.13 of [10]). *For any set $I \subseteq \mathbb{N}$,*

$$\text{HD}(J_I) = \inf\{t \geq 0 : P_I(t) \leq 0\}.$$

In particular, $\text{HD}(J_I)$ is the unique zero of P_I if such a zero exists.

We will need also the following theorem:

Theorem 3.2 (Theorem 2.1.5 of [10]). *Given $t \geq 0$, for every set $I \subseteq \mathbb{N}$ the function $I \mapsto P_I(t)$ is continuous in the following sense:*

$$P_I(t) = \lim_{N \rightarrow \infty} P_{I \cap \{1, \dots, N\}}(t).$$

3.3. Conformal measures. Conformal measures are an important tool for understanding the geometry of the limit set J_I . In many cases they coincide with either the normalized Hausdorff measure or the normalized packing measure.

Definition 3.3. Fix $t \geq 0$ and $I \subseteq \mathbb{N}$. A probability measure m on $[0, 1]$ is called *t-conformal* with respect to the iterated function system \mathcal{S}_I if $m(J_I) = 1$ and if

$$m(g_i(A)) = \int_A |g'_i|^t dm$$

for every Borel set $A \subseteq [0, 1]$ and for every $i \in I$.

Definition 3.4. Fix $I \subseteq \mathbb{N}$. The system \mathcal{S}_I is said to be *regular* if there exists $t \geq 0$ such that $P_I(t) = 0$.

Proposition 3.5 (Theorem 4.2.9 of [10]). *Fix $I \subseteq \mathbb{N}$. The following are equivalent:*

- (a) *The IFS \mathcal{S}_I is regular.*
- (b) *There exists a measure m and $t \geq 0$ such that m is t -conformal.*

Furthermore, in this case m and t are both unique, and $P(t) = 0$.

Corollary 3.6. *Fix $t \geq 0$ and $I \subseteq \mathbb{N}$. Then if $P(t) = 0$, then there exists a measure m which is t -conformal.*

Proof. Since $P(t) = 0$, it follows that the IFS \mathcal{S}_I is regular, so by Proposition 3.5, there exists a measure m and a number $t' \geq 0$ such that m is t' -conformal and $P(t') = 0$. But since P is strictly decreasing (part (b) of Proposition 4.2.8 of [10]), we have $t = t'$, so m is t -conformal. \square

Proposition 3.7. *Fix $I \subseteq \mathbb{N}$, and suppose that the IFS \mathcal{S}_I is regular. Let m_I be the unique conformal measure, and let $h = \text{HD}(J_I)$, so that m_I is h -conformal. Then there exists a unique Borel probability G -invariant measure μ_I on J_I absolutely continuous with respect to m_I . This measure is ergodic and equivalent to m_I . The logarithm of the corresponding Radon-Nikodym derivative is a bounded function on J_I .*

Proof. See Theorem 2.4.3 and part (c) of Corollary 2.7.5 in [10]. \square

Proposition 3.8.

- (a) *If the h_I -dimensional Hausdorff measure of J_I is positive (it is always finite), then the system \mathcal{S}_I is regular and we have*

$$m_I = \frac{\mathcal{H}^{h_I} \upharpoonright_{J_I}}{\mathcal{H}^{h_I}(J_I)}.$$

- (b) *If the h_I -dimensional packing measure of J_I is finite (it is always positive), then the system \mathcal{S}_I is regular and we have*

$$m_I = \frac{\mathcal{P}^{h_I} \upharpoonright_{J_I}}{\mathcal{P}^{h_I}(J_I)}.$$

Proof. An argument analogous to the proof of the change of variables formula demonstrates that both of the above expressions are h_I -conformal. The proposition therefore follows from Proposition 3.5. \square

3.4. Regularity properties of the IFS \mathcal{S}_I . Fix $I \subseteq \mathbb{N}$. In this subsection we discuss properties of the IFS \mathcal{S}_I that are stronger than just regularity.

Let

$$\theta_I := \inf\{t \geq 0 : P_I(t) < +\infty\}.$$

We have the following simple characterization of the number θ_I :

Proposition 3.9.

$$\theta_I = \inf\{t \geq 0 : \sum_{i \in I} \|g'_i\|_\infty^t < +\infty\} = \inf\{t \geq 0 : \sum_{i \in I} i^{-2t} < +\infty\}.$$

Proof. The first equation is part (a) of Proposition 4.2.8 of [10]; the second follows from the fact that $\|g'_i\|_\infty = i^{-2}$ for all $i \in \mathbb{N}$. \square

Definition 3.10. Fix $I \subseteq \mathbb{N}$. The system \mathcal{S}_I is said to be *strongly regular* if there exists $t \geq 0$ such that $0 < P_I(t) < +\infty$, and it is called *cofinitely regular* (or *hereditarily regular*) if $P_I(\theta_I) = +\infty$.

We have the following:

Proposition 3.11 (Theorem 4.3.5 of [10]).

- (a) *Every cofinitely regular system is strongly regular and every strongly regular system is regular.*
- (b) *For each strongly regular system \mathcal{S}_I , we have $\text{HD}(J_I) > \theta_I$.*

Proposition 3.12 (Theorem 4.3.4 of [10]). *Fix $I \subseteq \mathbb{N}$. The system \mathcal{S}_I is cofinitely regular if and only if the series*

$$\sum_{i \in I} \|g'_i\|_\infty^{\theta_I} = \sum_{i \in I} i^{-2\theta_I}$$

diverges.

Recall from Section 1 that if μ is a probability measure on $[0, 1]$ is invariant with respect to the Gauss map $G : [0, 1] \rightarrow [0, 1]$, then the integral

$$\chi_\mu(G) = \int \log |G'| d\mu$$

is called the *Lyapunov exponent* of the measure μ with respect to the Gauss map G . We have the following:

Proposition 3.13. *Fix $I \subseteq \mathbb{N}$. If the system \mathcal{S}_I is strongly regular, then $\chi_{\mu_I}(G) < +\infty$.*

Proof. By part (b) of Proposition 3.11, we have $h_I > \theta_I$. Fix $\theta_I < t < h_I$. Then by the definition of θ_I , the series

$$\sum_{i \in I} i^{-2t}$$

converges. Now we can estimate the Lyapunov exponent of μ_I as follows:

$$\begin{aligned} \int \log |G'| d\mu_I &\asymp \int \eta dm_I = \sum_{i \in I} \log(1+i) m_I(g_i([0, 1])) \\ &\asymp \sum_{i \in I} \log(1+i) i^{-2h_I} \lesssim \sum_{i \in I} i^{-2t} < \infty. \end{aligned}$$

□

3.5. Two lemmas. Each of the two lemmas in this subsection will be used several times throughout the remainder of the paper.

For every set $I \subseteq \mathbb{N}$ and for every $t \geq 0$ let

$$\lambda_t(I) = e^{P_I(t)}.$$

Lemma 3.14 (Lemma 4.3 of [4]). *Fix $\delta > 0$. Let $i \geq 2$ and let I be a finite subset of $\mathbb{N} \setminus \{i\}$. Then*

$$(3.1) \quad \lambda_\delta(I) + \left(\frac{1}{i+1}\right)^{2\delta} \leq \lambda_\delta(I \cup \{i\}) \leq \lambda_\delta(I) + \left(\frac{2}{i+2}\right)^{2\delta}.$$

Remark 3.15. Applying Theorem 3.2 to this lemma, we conclude that (3.1) holds in fact for all sets $I \subseteq \mathbb{N} \setminus \{i\}$.

Recall that we have defined $\xi(x) = \lfloor 1/x \rfloor$ to be the first entry of the continued fraction expansion of x . For any $\omega \in \mathbb{N}^n$, let

$$S_\omega := g_\omega([0, 1]) = \{x \in [0, 1] : \xi(G^j(x)) = \omega_j \quad \forall j = 0, \dots, n-1\},$$

i.e. S_ω is the set of all numbers whose continued fraction expansions begin with the sequence $\omega_0, \dots, \omega_{n-1}$. Furthermore, for each $k \in \mathbb{N}$ let

$$S_{\omega, k}^+ = \bigcup_{i \leq k} S_{\omega i}.$$

Lemma 3.16. *Fix $I \subseteq \mathbb{N}$, and suppose that the IFS \mathcal{S}_I is regular. Let h_I be the Hausdorff dimension of J_I and let m_I be the unique h_I -conformal measure of \mathcal{S}_I . Then*

$$(3.2) \quad \frac{m_I(S_{\omega i})}{m_I(S_{\omega})} \geq \frac{1}{4^{h_I}} i^{-2h_I}$$

and

$$(3.3) \quad \frac{m_I(S_{\omega, k}^+)}{m_I(S_{\omega})} \leq 1 - \frac{1}{4^{h_I}} \sum_{\substack{i \in I \\ i > k}} i^{-2h_I}.$$

Proof. It is clear that (3.3) follows from (3.2). To demonstrate (3.2), note that since m_I is h_I -conformal we have

$$\begin{aligned} \frac{m_I(S_{\omega i})}{m_I(S_{\omega})} &= \frac{m_I(g_{\omega i}([0, 1]))}{m_I(g_{\omega}([0, 1]))} = \frac{\int |(g'_{\omega i}(x))|^{h_I} dm_I(x)}{\int |g'_{\omega}(x)|^{h_I} dm_I(x)} \\ &\geq \frac{\min_{[0, 1]} |g'_{\omega}|^{h_I}}{\max_{[0, 1]} |g'_{\omega}|^{h_I}} \min_{[0, 1]} |g'_i|^{h_I}. \end{aligned}$$

On the other hand, we have (see [9], line -10 of p.4997, or by direct computation)

$$\frac{\max_{[0, 1]} |g'_{\omega}|}{\min_{[0, 1]} |g'_{\omega}|} \leq 4$$

which yields (3.2). \square

4. EXTREMALITY OF CONFORMAL MEASURES

Fix $I \subseteq \mathbb{N}$, and suppose that the IFS \mathcal{S}_I is regular. In this section we discuss the extremality of the measures m_I and μ_I defined in Section 3. Note that since m_I and μ_I are absolutely continuous to each other, m_I is extremal if and only if μ_I is.

By Theorem 2.1, if $\chi_{\mu_I} < +\infty$, then μ_I is extremal. By Proposition 3.13, if \mathcal{S}_I is strongly regular, then $\chi_{\mu_I} < \infty$. The following proposition gives very general sufficient conditions for \mathcal{S}_I to be strongly regular:

Proposition 4.1. *If $I \subseteq \mathbb{N}$, then any of the following three conditions entail strong regularity of the iterated function system \mathcal{S}_I , and thus the extremality of the measures m_I and μ_I :*

- (a) I is finite.
- (b) The series $\sum_{a \in I} a^{-2\theta_I}$ diverges.
- (c) The Hausdorff dimension of the limit set of the IFS is strictly greater than $1/2$.
- (d) $1, 2 \in I$.

Proof. Item (a) follows directly from the definition (we have $0 < P_I(0) < \infty$); item (b) follows from (a) of Proposition 3.11 and Proposition 3.12; item (c) follows from Theorem 4.3.10 of [10] along with the observation that $\theta_I \leq \theta_{\mathbb{N}} = 1/2$; item (d) follows from item (c) and the fact, proven in [2], that $h_{\{1,2\}} = \text{HD}(J_{\{1,2\}}) > 1/2$. \square

Remark 4.2. In case (a), the extremality of μ_I is obvious since its topological support J_I is contained in the set of badly approximable numbers (see Remark 2.8).

Remark 4.3. The main result of [11], namely, the extremality part of Theorem 4.5 of that paper, can be deduced from part (b) of Proposition 4.1.

Example 4.4. Fix $a \geq 2$ and let I be the geometric series $I = \{a, a^2, \dots\}$. Then condition (b) of Proposition 4.1 is satisfied. Thus the measure μ_I is extremal. On the other hand, μ_I is not absolutely decaying (see below), so the extremality of μ_I does not follow from Weiss's theorem [12].

Proof that μ_I is not absolutely decaying. Fix $n \in \mathbb{N}$, and let $x_n = a^{-n} \in \overline{J_I}$. Then

$$B\left(x_n, \frac{1}{a^n} - \frac{1}{a^{n+1}}\right) \cap J_I = B\left(x_n, \frac{1}{a^n} - \frac{1}{a^{n+1}}\right) \cap J_I.$$

If μ were absolutely α -decaying, we would therefore have

$$\begin{aligned} 1 &= \frac{\mu\left[B\left(x_n, \frac{1}{a^n} - \frac{1}{a^{n+1}}\right)\right]}{\mu\left[B\left(x_n, \frac{1}{a^n} - \frac{1}{a^{n+1}}\right)\right]} \leq C \left(\frac{\frac{1}{a^n} - \frac{1}{a^{n+1}}}{\frac{1}{a^n} - \frac{1}{a^{n+1}}}\right)^\alpha \\ &\asymp \left(\frac{1/a^{2n}}{1/a^n}\right)^\alpha = \frac{1}{a^{n\alpha}}, \end{aligned}$$

which is a contradiction for n large enough. \square

The remainder of this section will be devoted to proving the following theorem:

Theorem 4.5. *There exists a measure μ invariant with respect to the Gauss map which gives full measure to the Liouville numbers. In particular, μ is not extremal.*

The measure μ will be of the form μ_I for some $I \subseteq \mathbb{N}$ defining a regular system \mathcal{S}_I .

Define a sequence of finite subsets $I_N \subseteq \mathbb{N}$ recursively in the following manner:

1. Let $I_0 = \emptyset$.

2. Suppose that the set I_{N-1} has been defined. Let $M_{N-1} = \max(I_{N-1})$. (By convention let $\max(\emptyset) = 0$.)
3. Choose $m_N \in \mathbb{N}$ large enough so that:

$$\log(1 + m_N) \geq N 4^N \log(1 + M_{N-1})$$

$$\left(\frac{2}{m_N + 2} \right)^{2\delta} \leq 2^{-N}.$$

4. Let $R_N \subseteq \{m_N, \dots\}$ be a finite set satisfying:

$$(4.1) \quad 1 - 2^{-(N-1)} \leq \lambda_\delta(I_{N-1} \cup R_N) < 1 - 2^{-N}.$$

(The existence of such a set R_N is verified below.)

5. Let $I_N = I_{N-1} \cup R_N$ and then go back to step 2.

We now check that in step 4, it is always possible to find a set R_N which satisfies (4.1). We first claim that

$$(4.2) \quad \lambda_\delta(I_{N-1}) < 1 - 2^{-N} < \lambda_\delta(I_{N-1} \cup \{m_N, \dots\}).$$

Indeed, the left inequality follows from the induction hypothesis (or by direct computation in the case $N = 1$). The right hand side follows from Lemma 3.14 and the fact that the series

$$\sum_{i=m_N}^{\infty} \left(\frac{1}{i+1} \right)^{2\delta}$$

diverges (since $\delta \leq 1/2$).

It follows from (4.2) that there exists $K \in \{m_N, \dots\}$ so that

$$\begin{aligned} \lambda_\delta(I_{N-1} \cup \{m_N, \dots, K\}) &< 1 - 2^{-N} \\ &\leq \lambda_\delta(I_{N-1} \cup \{m_N, \dots, K+1\}). \end{aligned}$$

Let $R_N = \{m_N, \dots, K\}$. By Lemma 3.14, we have

$$\begin{aligned} \lambda_\delta(I_{N-1} \cup \{m_N, \dots, K\}) &\geq \\ &\geq \lambda_\delta(I_{N-1} \cup \{m_N, \dots, K+1\}) - \left(\frac{2}{(K+1)+2} \right)^{2\delta} \\ &\geq 1 - 2^{-N} - 2^{-N} = 1 - 2^{-(N-1)} \end{aligned}$$

which demonstrates (4.1).

Let

$$I = \bigcup_N I_N.$$

By Theorem 3.2, we have $\lambda_\delta(I) = 1$, and thus $P_I(\delta) = 0$. By Corollary 3.6 and Proposition 3.7 we have that $\text{HD}(J_I) = \delta$, and that there exists a δ -conformal measure m_I and an absolutely continuous G -invariant measure μ_I . To complete the proof we need to show that

m_I , and thus μ_I , gives full measure to the set of Liouville numbers. To this end, fix $N \in \mathbb{N}$. By Lemma 3.14 we have

$$(4.3) \quad 1 - \lambda_\delta(I \cap \{1, \dots, M_N\}) \leq \sum_{\substack{i \in I \\ i > M_N}} \left(\frac{2}{2+i} \right)^{2\delta} \leq 4^\delta \sum_{\substack{i \in I \\ i > M_N}} \left(\frac{1}{1+i} \right)^{2\delta}.$$

Fix $\omega = (\omega_j)_{j=0}^{n-1} \in \mathbb{N}^n$. It then follows from (3.3) and (4.3) that

$$\frac{m_I(S_{\omega, M_N}^+)}{m_I(S_\omega)} \leq 1 - \frac{1}{16^\delta} (1 - \lambda_\delta(I \cap \{1, \dots, M_N\})),$$

where S_ω and $S_{\omega, k}^+$ are defined as in Lemma 3.16. Invoking (4.1) gives

$$(4.4) \quad \frac{m_I(S_{\omega, M_N}^+)}{m_I(S_\omega)} \leq 1 - c2^{-N},$$

where $c = 1/16^\delta$. Now for each $n \in \mathbb{N}$ let

$$S_{n, N} = \{x \in [0, 1] : \xi(G^j(x)) \leq M_N \quad \forall j = 0, \dots, n-1\}.$$

Formula (4.4) yields

$$\frac{m_I(S_{n+1, N})}{m_I(S_{n, N})} \leq 1 - c2^{-N}.$$

Iterating yields

$$m_I(S_{n, N}) \leq (1 - c2^{-N})^n.$$

Letting $n = 4^N$, we see that

$$m_I(S_{4^N, N}) \leq e^{-c2^N}$$

and thus

$$\sum_{N=0}^{\infty} m_I(S_{4^N, N}) < \infty.$$

Thus by the Borel-Cantelli lemma, m_I -almost every point $x \in J_I$ lies in only finitely many sets of the form $S_{4^N, N}$. Fix such a point x , and we will show that x is a Liouville number. By Corollary 2.7, it suffices to demonstrate that for all $c > 0$ we have

$$(4.5) \quad \eta(G^n(x)) \geq c \sum_{j=0}^{n-1} \eta(G^j(x))$$

for infinitely many $n \in \mathbb{N}$, where η is defined as in Definition 2.5. Indeed, for all but finitely many $N \in \mathbb{N}$, we have $x \notin S_{4^N, N}$ and so there exists $n \leq 4^N$ such that $\xi(G^n(x)) > M_N$. Without loss of generality, we may assume that n is minimal with this property, i.e.

$\xi(G^j(x)) \leq M_N$ for all $j < n$. Now, since I does not contain any numbers between M_N and m_{N+1} , we have $\xi(G^n(x)) \geq m_{N+1}$ and thus

$$\begin{aligned} \eta(G^n(x)) &\geq \log(1 + m_{N+1}) = N4^N \log(1 + M_N) \\ &\geq N \sum_{j=0}^{n-1} \log(1 + \xi(G^j(x))) \\ &= N \sum_{j=0}^{n-1} \eta(G^j(x)) \end{aligned}$$

which demonstrates that (4.5) has infinitely many solutions. Thus the proof of Theorem 4.5 is complete.

5. COMBINATORIAL CHARACTERIZATIONS OF AHLFORS REGULARITY

In this section we prove Theorem 5.5 which gives a combinatorial characterization of Ahlfors regularity of J_I . We begin by recalling the following theorems:

Theorem 5.1 (Theorem 4.1 of [9]). *Fix a set $I \subseteq \mathbb{N}$, and suppose that the IFS \mathcal{S}_I is regular. Let $h = \text{HD}(J_I)$, and let m_I be an h -conformal measure. Then the following are equivalent:*

- (a) $\mathcal{H}^h(J_I) > 0$.
- (b)

$$(5.1) \quad \sup_{k_1 < k_2} \frac{(k_1 k_2)^h}{(k_2 - k_1)^h} \sum_{\substack{i \in I \\ k_1 \leq i \leq k_2}} i^{-2h} < \infty.$$

- (c) m_I is Ahlfors h -lower regular i.e.

$$m_I(B(x, r)) \lesssim r^h \quad \forall x \in J_I \quad \forall r \leq 1.$$

Proof. The equivalence of (a) and (b) is proven in Theorem 4.1 of [9]. The implication (a) \Rightarrow (c) follows from the last line of the proof of the implication (c) \Rightarrow (a) of Theorem 4.5.3 of [10] (just before the mass distribution principle is applied), and the implication (c) \Rightarrow (a) is the mass distribution principle. \square

Theorem 5.2 (Theorem 5.1 of [9]). *Fix an infinite set $I \subseteq \mathbb{N}$, and suppose that the IFS \mathcal{S}_I is regular. Let $h = \text{HD}(J_I)$, and let m_I be an h -conformal measure. Then the following are equivalent:*

- (a) $\mathcal{P}^h(J_I) < \infty$.

(b) *Both of the following hold:*

$$\inf_{\substack{k_1 < k_2 \\ B\left(\frac{2k_1 k_2}{k_1 + k_2}, 1\right) \cap I \neq \emptyset}} \frac{(k_1 k_2)^h}{(k_2 - k_1)^h} \sum_{\substack{i \in I \\ k_1 \leq i \leq k_2}} i^{-2h} > 0$$

$$\inf_{k \geq 1} k^h \sum_{\substack{i \in I \\ i \geq k}} i^{-2h} > 0.$$

(c) m_I is Ahlfors h -upper regular i.e.

$$m_I(B(x, r)) \gtrsim r^h \quad \forall x \in J_I \quad \forall r \leq 1.$$

Note that the assumption that I is infinite is necessary in this theorem since any finite IFS satisfies (a) and (c) but not (b).

Proof. The equivalence of (a) and (b) is proven in Theorem 5.1 of [9]. The implication (a) \Rightarrow (c) follows from the last line of the proof of the implication (c) \Rightarrow (a) of Theorem 4.5.5 of [10] (just before the mass distribution principle for packing measure is applied), and the implication (c) \Rightarrow (a) is the mass distribution principle for packing measure. \square

We can add new equivalences to Theorems 5.1 and 5.2 as follows:

Theorem 5.3. (a)-(c) of Theorem 5.1 are equivalent to the following:

(d) *Both (i) and (ii) hold:*

(i) *For all $y \in \mathbb{N}$ and for all $1 \leq r \leq y/2$,*

$$\#(B(y, r) \cap I) \lesssim r^h.$$

(ii) *For all $k \in \mathbb{N}$,*

$$\sum_{\substack{i \in I \\ i > k}} i^{-2h} \lesssim k^{-h}.$$

Theorem 5.4. (a)-(c) of Theorem 5.2 are equivalent to the following:

(d) *Both (i) and (ii) hold:*

(i) *For all $y \in I$ and for all $1 \leq r \leq y/2$,*

$$\#(B(y, r) \cap I) \gtrsim r^h.$$

(ii) *For all $k \in \mathbb{N}$,*

$$\sum_{\substack{i \in I \\ i > k}} i^{-2h} \gtrsim k^{-h}.$$

Proof of Theorems 5.3 and 5.4. By way of illustration we shall show that (d) of Theorem 5.3 implies (b) of Theorem 5.1. The proof of the other implications are left to the reader.

Fix $k_1 < k_2$. If $I \cap [k_1, k_2] = \emptyset$, then the pair (k_1, k_2) does not contribute to the supremum (5.1). Thus, suppose that $I \cap [k_1, k_2] \neq \emptyset$, and fix $y \in I \cap [k_1, k_2]$. Let $r = \max(k_2 - y, y - k_1)$. If $r \leq y/2$, then we have

$$\begin{aligned} \frac{2}{3}k_2 &\leq y \leq 2k_1 \\ r &\leq k_2 - k_1 \leq 2r \end{aligned}$$

and thus by (d)(i) of Theorem 5.3 we have

$$\begin{aligned} \frac{(k_1 k_2)^h}{(k_2 - k_1)^h} \sum_{\substack{i \in I \\ k_1 \leq i \leq k_2}} i^{-2h} &\asymp \frac{(y^2)^h}{r^h} \sum_{\substack{i \in I \\ k_1 \leq i \leq k_2}} y^{-2h} \\ &= r^{-h} \#(I \cap [k_1, k_2]) \\ &\leq r^{-h} \#(I \cap B(y, r)) \lesssim r^{-h} r^h = 1. \end{aligned}$$

On the other hand, suppose that $r \geq y/2$. Then

$$\begin{aligned} k_2 - k_1 &\geq r \geq \frac{k_1}{2} \\ k_2 &\geq \frac{3}{2}k_1 \\ k_2 - k_1 &\geq \frac{k_2}{3} \end{aligned}$$

and thus by (d)(ii) of Theorem 5.3 we have

$$\begin{aligned} \frac{(k_1 k_2)^h}{(k_2 - k_1)^h} \sum_{\substack{i \in I \\ k_1 \leq i \leq k_2}} i^{-2h} &\leq \frac{(k_1 k_2)^h}{(k_2/3)^h} \sum_{\substack{i \in I \\ i \geq k_1}} i^{-2h} \\ &\asymp k_1^h \sum_{\substack{i \in I \\ i \geq k_1}} i^{-2h} \lesssim k_1^h k_1^{-h} = 1 \end{aligned}$$

Thus either way we have

$$\frac{(k_1 k_2)^h}{(k_2 - k_1)^h} \sum_{\substack{i \in I \\ k_1 \leq i \leq k_2}} i^{-2h} \lesssim 1,$$

which is equivalent to (5.1). \square

If we restrict our attention to sets I which satisfy both the conditions of Theorem 5.1 and those of Theorem 5.2, then we get even more characterizations:

Theorem 5.5. *Fix an infinite set $I \subseteq \mathbb{N}$, and let $h = h_I = \text{HD}(J_I)$. Then (a)-(c3) are equivalent and imply (d)-(e):*

- (a) $\mathcal{H}^h(J_I) > 0$ and $\mathcal{P}^h(J_I) < \infty$.
 - (b1) $\mathcal{H}^h \upharpoonright_{J_I}$ is Ahlfors h -regular.
 - (b2) $\mathcal{P}^h \upharpoonright_{J_I}$ is Ahlfors h -regular.
 - (b3) The IFS \mathcal{S}_I is regular and m_I is Ahlfors h -regular.
 - (b4) The IFS \mathcal{S}_I is regular and μ_I is Ahlfors h -regular.
 - (c1) For all $y \in I$ and $r \geq 1$
- (5.2) $\#(B(y, r) \cap I) \asymp r^h.$
- (c2) Both of the following hold:
 - (i) (5.2) holds for all $y \in I$ and $1 \leq r \leq y/2$.
 - (ii) There exists $m \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, we have $[k, mk] \cap I \neq \emptyset$.
 - (c3) Both of the following hold:
 - (i) (5.2) holds for all $y \in I$ and $1 \leq r \leq y/2$.
 - (ii) For all $k \in \mathbb{N}$ we have
- (5.3) $\sum_{\substack{i \in I \\ i > k}} i^{-2h} \asymp k^{-h}.$
- (d) $\theta_I = h/2$.
 - (e) The IFS \mathcal{S}_I is cofinitely regular.

Proof. Let us first assume that \mathcal{S}_I is regular. Then the equivalence of (a), (b3), and (c3) follows directly from Theorems 5.3 and 5.4. The equivalence of (b3) and (b4) follows from part Proposition 3.7. To see that (b1) and (b3) are equivalent, note that by part (a) of Proposition 3.8, if the equivalence fails then $\mathcal{H}^h(J_I) = 0$. But in this case, clearly (b1) and (a) are both false, so since (a) is equivalent to (b3) we have (b1) \Leftrightarrow (b3). A similar argument yields the equivalence of (b2) and (b3).

We next show that (c1) \Leftrightarrow (c2) \Leftrightarrow (c3) \Rightarrow (d), (e). In these proofs we do not assume regularity of \mathcal{S}_I .

Proof of (c3) \Rightarrow (c2). Suppose that (c3) holds. Let C be the implied constant of (5.3), and let $m = \lceil C^{2/h} \rceil + 1$. Then for any $k \in \mathbb{N}$, we have

$$\sum_{\substack{i \in I \\ i > mk}} i^{-2h} \leq C(mk)^{-h} < C^{-1}k^{-h} \leq \sum_{\substack{i \in I \\ i > k}} i^{-2h}$$

which demonstrates that $[k, mk] \cap I \neq \emptyset$. \triangleleft

Proof of (c2) \Rightarrow (c1), (c3), (d), (e). Suppose that (c2) holds. We claim that

$$(5.4) \quad \#(I \cap [k, 3mk]) \asymp k^h$$

for all $k \in \mathbb{N}$. Indeed, the upper bound can be achieved by covering $I \cap [k, 3mk]$ by finitely many sets of the form $B(y, y/2)$, where $y \in I \cap [k, 3mk]$, and applying (5.2). The lower bound follows from choosing a point $y \in I \cap [2k, 2mk]$ and applying (5.2) to the set $B(y, y/2)$.

From (5.4), we calculate that for any $t \geq 0$ and $k \in \mathbb{N}$ we have

$$\begin{aligned} \sum_{\substack{i \in I \\ i > k}} i^{-2t} &\asymp \sum_{n \in \mathbb{N}} \sum_{\substack{i \in I \\ (3m)^n k < i \leq (3m)^{n+1} k}} i^{-2t} \\ &\asymp \sum_{n \in \mathbb{N}} [(3m)^n k]^h [(3m)^n k]^{-2t} \end{aligned}$$

which diverges if $t \leq h/2$ and is otherwise asymptotic to

$$k^{h-2t} < \infty.$$

Specializing to the case $t = h$ yields (c3). Applying Proposition 3.9 yields (d). Finally, part Proposition 3.7 yields (e).

To demonstrate (c1), fix $y \in I$ and $r \geq 1$. If $r \leq y/2$, then we have (5.2) for free. Thus, suppose $r > y/2$. Let $N = \lceil \log_{3m}(y + r) \rceil$. Then

$$B(y, r) \subseteq \bigcup_{n=0}^N [(3m)^n, (3m)^{n+1}].$$

On the other hand, for each $n \leq N$ we have from (5.4)

$$\#(I \cap [(3m)^n, (3m)^{n+1}]) \asymp [(3m)^n]^h$$

and summing yields

$$\#(B(y, r) \cap I) \lesssim [(3m)^N]^h \asymp r^h.$$

To get the lower bound, note that

$$\#(B(y, r) \cap I) \geq \#(B(y, y/2) \cap I) \asymp (y/2)^h \asymp y^h.$$

This bound is good enough unless $r \geq y$. In this case, let $k = \lfloor r/(3m) \rfloor$, and (5.4) yields the bound. \triangleleft

Proof of (c1) \Rightarrow (c2). Similar to the proof of (c3) \Rightarrow (c2). \triangleleft

This completes the proof of the theorem in the case where \mathcal{S}_I is regular. Suppose on the other hand that \mathcal{S}_I is not regular. Then (b3) and (b4) are clearly false. Applying parts (a) and (b) of Proposition 3.8 yields that (a), (b1), and (b2) are false. Applying part (a) of Proposition 3.11 yields that (d) is false. Since we have $(c1) \Leftrightarrow (c2) \Leftrightarrow (c3) \Rightarrow (d)$, and the proof of this did not depend on the regularity of \mathcal{S}_I , we have that (c1)-(c3) are false. This yields the theorem. \square

6. PROOF OF THEOREM 6.1

In this section we will prove the following theorem:

Theorem 6.1. *Fix an infinite set $I \subseteq \mathbb{N}$, and let h be the Hausdorff dimension of J_I . Assume that the h -dimensional Hausdorff measure \mathcal{H}^h restricted to J_I is Ahlfors h -regular. Let $\mu = \mathcal{H}^h \upharpoonright_{J_I}$, and let $\psi : \mathbb{N} \rightarrow (0, +\infty)$ be a function such that the function $q \mapsto q^2\psi(q)$ is nonincreasing. Then μ -almost every (resp. μ -almost no) point is ψ -approximable, assuming that the series*

$$(6.1) \quad \sum_{q=1}^{\infty} q^{2\alpha-1}\psi(q)^{\alpha}$$

diverges (resp. converges).

As noted in the introduction, the convergence case follows from Weiss's theorem [12].

Fix a function $\psi : \mathbb{N} \rightarrow (0, \infty)$, and suppose that the series (6.1) diverges and that the function $q \mapsto q^2\psi(q)$ is nonincreasing. By (a) of Proposition 3.8, we have $m_I \asymp \mathcal{H}^h \upharpoonright_{J_I}$, so to prove the theorem it suffices to show that m_I -almost every number is ψ -approximable. In fact, we will demonstrate the (slightly) stronger statement that m_I -almost every number is ψ -well approximable.

By (b1) \Rightarrow (e) of Theorem 5.5, the iterated function system $\mathcal{S}_I = \{g_a\}_{a \in I}$ is cofinitely regular. Thus the Lyapunov exponent of μ_I is finite (Proposition 3.13 and part (a) of Proposition 3.11) and in particular $0 < \int \eta d\mu_I < \infty$ (see (2.7)), where η is defined as in Definition 2.5. Thus by the Birkhoff ergodic theorem, we have

$$\frac{1}{n} \sum_{j=0}^{n-1} \eta(G^j(x)) \xrightarrow[n]{\rightarrow} E := \int \eta d\mu_I$$

for μ_I -almost every $x \in [0, 1]$. Combining the above equation with (2.6) gives

$$(6.2) \quad \frac{E}{2} \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log(q_n) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log(q_n) \leq E.$$

Let $x \in [0, 1]$ be a point such that (6.2) holds but which is not ψ -well approximable. By Lemma 2.4 there exists $K > 0$ such that for all $n \in \mathbb{N}$, (2.1) fails to hold. Combining (6.2), the negation of (2.1), and the fact that $\phi(q) = \frac{1}{q^2\psi(q)}$ is nondecreasing yields

$$(6.3) \quad \omega_n = \xi(G^n(x)) \leq K\phi(\gamma^n)$$

for all n sufficiently large, where $\gamma = 1 + \lceil e^E \rceil$. By increasing K , we may ensure that (6.3) holds for all $n \in \mathbb{N}$.

Thus, we are done if we show that the set of x for which there exists K such that (6.3) holds for all $n \in \mathbb{N}$ is a null set. Given $n \in \mathbb{N}$ and $K > 0$ let

$$S_{\psi,n,K} = \{x \in J_I : (6.3) \text{ holds for } n, K\}$$

and

$$S_{\psi,n,K}^+ = \bigcap_{j=0}^{n-1} S_{\psi,j,K}.$$

To complete the proof of Theorem 6.1 we must therefore show that

$$(6.4) \quad m_I(S_{\psi,\infty,K}^+) = 0 \quad \forall K > 0.$$

Fix $K > 0$.

For each $n \in \mathbb{N}$, let $k_n = K\phi(\gamma^n)$. In the notation of Lemma 3.16, we have

$$\begin{aligned} S_{\psi,n,K} &= \bigcup_{\omega \in A_n} S_\omega \\ S_{\psi,n+1,K} &= \bigcup_{\omega \in A_n} S_{\omega,k_n}^+, \end{aligned}$$

where

$$A_n = \prod_{j=0}^{n-1} \{1, \dots, k_j\}.$$

It therefore follows from (3.3) that

$$\frac{m_I(S_{\psi,n+1}^+)}{m_I(S_{\psi,n}^+)} \leq 1 - \frac{1}{4^{h_I}} \sum_{\substack{i \in I \\ i > k_n}} i^{-2h_I}.$$

On the other hand, by the implication (b1) \Rightarrow (c3) of Theorem 5.5 we have

$$(6.5) \quad \sum_{\substack{i \in I \\ i > k_n}} i^{-2h} \asymp k_n^{-h} \asymp \phi(\gamma^n)^{-h}.$$

Thus for some constant $K_2 > 0$ depending on K , we have

$$\frac{m_I(S_{\psi, n+1, K}^+)}{m_I(S_{\psi, n, K}^+)} \leq 1 - K_2 \phi(\gamma^n)^{-h}.$$

Thus

$$m_I(S_{\psi, \infty, K}^+) \leq \prod_{n=0}^{\infty} (1 - K_2 \phi(\gamma^n)^{-h})$$

which is zero if the series

$$(6.6) \quad \sum_{n=0}^{\infty} \phi(\gamma^n)^{-h}$$

diverges. Now, by Cauchy's condensation test, (6.6) diverges if and only if (6.1) diverges. This demonstrates (6.4), completing the proof.

7. PROOF OF THEOREM 7.1

In this section we will prove the following theorem:

Theorem 7.1. *For every $0 < \delta \leq 1$ there exists an infinite set $I \subseteq \mathbb{N}$ such that $\text{HD}(J_I) = \delta$ and such that $\mathcal{H}^\delta \upharpoonright_{J_I}$ is Ahlfors δ -regular.*

Fix $0 < \delta \leq 1$. If $\delta = 1$, we let $I = \mathbb{N}$; the conclusion of the proposition is satisfied since then $\mathcal{H}^\delta \upharpoonright_{J_I}$ is simply Lebesgue measure. Thus, we shall assume without loss of generality that $\delta < 1$. We observe that by the implication (c1) \Rightarrow (b1) of Theorem 5.5, to prove Theorem 7.1 it suffices to find a set I satisfying

$$(7.1) \quad \text{HD}(J_I) = \delta$$

and

$$(7.2) \quad \#(B(y, r) \cap I) \asymp r^\delta.$$

We will begin by finding a set I_0 which satisfies (7.2) but not (7.1). Then we will construct a set R which satisfies (7.1) but not (7.2). Finally we will combine I_0 and R into a single set I_δ which satisfies both (7.1) and (7.2).

7.1. Constructing I_0 .

Lemma 7.2. *There exists a set $I_0 \subseteq \mathbb{N}$ satisfying (7.2).*

Proof. Indeed, let I_0 be the set of all sums of the form

$$1 + \sum_{n \in \mathbb{N}} a_n \lfloor 2^{n/\delta} \rfloor,$$

where $a_n = 0$ or 1 for all $n \in \mathbb{N}$, with only finitely many 1s. It is readily verified that I_0 satisfies (7.2). \square

7.2. Constructing R . We define a sequence of subsets $R_N \subseteq \mathbb{N}$ by induction:

1. Let $R_1 = \{1\}$.
2. Suppose that $R_{N-1} \subseteq \{1, \dots, N-1\}$ has been defined for some $N \geq 2$. If

$$\lambda_\delta(R_{N-1} \cup \{N\}) < 1,$$

then let $R_N = R_{N-1} \cup \{N\}$, otherwise let $R_N = R_{N-1}$.

Observation 7.3. For all $N \in \mathbb{N}$

$$\lambda_\delta(R_N) < 1.$$

Proof. The base case follows either from direct computation or from Bowen's formula (Theorem 3.1); the inductive step follows from the construction of R_N . \square

Claim 7.4. $R := \bigcup_N R_N$ is not cofinite.

Proof. By Theorem 3.2 and by the previous observation, we have $\lambda_\delta(R) \leq 1$. Combining with Bowen's formula, we see that $\text{HD}(J_R) \leq \delta < 1 = \text{HD}(J_{\mathbb{N}})$. In particular, $R \neq \mathbb{N}$.

Thus if we suppose by contradiction that R is cofinite, then $\mathbb{N} \setminus R$ has a maximal element M ; moreover, we know that $M \geq 2$ since $1 \in R$. But then by the construction of R_M , we have

$$\lambda_\delta(R_{M-1} \cup \{M\}) \geq 1$$

and so by Lemma 3.14 we have

$$(7.3) \quad \lambda_\delta(R_{M-1}) \geq 1 - \left(\frac{2}{2+M} \right)^{2\delta}.$$

On the other hand, by Observation 7.3 we have

$$\lambda_\delta(R_{M-1} \cup \{M+1, \dots, N\}) = \lambda_\delta(R_N) < 1$$

for every $N \in \mathbb{N}$. So, applying Lemma 3.14, we see that

$$\begin{aligned} \left(\frac{2}{2+M} \right)^{2\delta} &> \lambda_\delta(R_{M-1} \cup \{M+1, \dots, N\}) - \lambda_\delta(R_{M-1}) \\ &\geq \sum_{i=M+1}^N \left(\frac{1}{1+i} \right)^{2\delta} \\ &> \int_{x=M+1}^{N+1} \left(\frac{1}{1+x} \right)^{2\delta} dx. \end{aligned}$$

Since N was arbitrary, we can take the limit as N approaches infinity and so we have

$$\int_{x=M+1}^{\infty} \left(\frac{1}{1+x} \right)^{2\delta} dx < \left(\frac{2}{2+M} \right)^{2\delta}.$$

If $\delta \leq 1/2$, then the left hand integral diverges, a contradiction. If $\delta > 1/2$, the left hand integral converges and we have

$$\frac{(M+2)^{1-2\delta}}{2\delta-1} < \left(\frac{2}{2+M} \right)^{2\delta}.$$

Rearranging yields

$$M+2 < 2^{2\delta}(2\delta-1) \leq 2^2(2-1) = 4.$$

This contradicts $M \geq 2$ and the proof is finished. \square

Observation 7.5. It follows from (7.3), Observation 7.3, and Theorem 3.2 that

$$\text{HD}(J_R) = \delta.$$

7.3. Combining I_0 and R . Fix $N_1 \in \mathbb{N} \setminus R$ large, to be determined later.⁶ By the construction of R_{N_1} we have (7.3) with $M = N_1$, and so

$$1 - \left(\frac{2}{2+N_1} \right)^{2\delta} \leq \lambda_\delta(R_{N_1-1}) < 1.$$

Now let I_0 be as in Lemma 7.2, and let

$$\begin{aligned} I_+ &:= 2I_0 \\ I_- &:= 2I_0 - 1. \end{aligned}$$

It is evident that any set $I_\delta \subseteq \mathbb{N}$ satisfying

$$(7.4) \quad I_- \subseteq_* I_\delta \subseteq_* I_+ \cup I_-$$

satisfies (7.2), where $A \subseteq_* B$ means $\#(A \setminus B) < \infty$. We will construct such a set recursively. Now by the implication (c1) \Rightarrow (c3) of Theorem 5.5⁷, we have

$$\sum_{i \in I_-} \left(\frac{2}{2+i} \right)^{2\delta} \asymp \sum_{i \in I_-} i^{-2\delta} < \infty;$$

thus we may choose N_2 large enough so that

$$(7.5) \quad \sum_{i \in I_- \setminus \{1, \dots, N_2\}} \left(\frac{2}{2+i} \right)^{2\delta} < 1 - \lambda_\delta(R_{N_1-1}).$$

⁶Specifically, we let N_1 be large enough so that (7.7) cannot hold whenever $M \geq N_1$.

⁷Note that the implication holds even when $h \neq \text{HD}(J_I)$.

We will now construct a sequence of sets $(I_N)_{N \geq N_1-1}$ recursively in the following manner:

1. Let

$$I_{N_1-1} = R_{N_1-1} \cup (I_- \setminus \{1, \dots, N_2\}).$$

2. Suppose that I_{N-1} has been defined for some $N \geq N_1$. If $N \notin I_+ \cup \{N_1\}$, then let $I_N = I_{N-1}$.

3. If $N \in I_+ \cup \{N_1\}$, and if

$$\lambda_\delta(I_{N-1} \cup \{N\}) < 1,$$

then let $I_N = I_{N-1} \cup \{N\}$.

4. Otherwise, let $I_N = I_{N-1}$.

Observation 7.6. For all $N \geq N_1 - 1$

$$\lambda_\delta(I_N) < 1.$$

Proof. The base case of induction follows from Lemma 3.14 together with (7.7). The induction step follows from the construction of I_N . \square

Claim 7.7. *Case 4 occurs infinitely many times.*

Proof. As $N_1 \notin R$, we know that Case 4 occurs at least once, namely at $N = N_1$. If we suppose by contradiction that it occurs only finitely often, then there is some maximal value M at which it occurs. In particular

$$\lambda_\delta(I_{M-1} \cup \{M\}) \geq 1,$$

and applying Lemma 3.14 gives

$$(7.6) \quad \lambda_\delta(I_{M-1}) \geq 1 - \left(\frac{2}{2+M} \right)^{2\delta}.$$

On the other hand, by the above observation and by the maximality of M we have

$$\lambda_\delta(I_{M-1} \cup (I_+ \cap \{M+1, \dots, N\})) < 1$$

for all $N \in \mathbb{N}$. Combining these last two formulas and then applying Lemma 3.14, we see that

$$\begin{aligned} \left(\frac{2}{2+M} \right)^{2\delta} &> \lambda_\delta(I_{M-1} \cup (I_+ \cap \{M+1, \dots, N\})) - \lambda_\delta(I_{M-1}) \\ &\geq \sum_{\substack{i=M+1 \\ i \in I_+}}^N \left(\frac{1}{1+i} \right)^{2\delta}. \end{aligned}$$

Since N was arbitrary, we can take the limit as N approaches infinity which yields

$$(7.7) \quad M^{-2\delta} \asymp \left(\frac{2}{2+M} \right)^{2\delta} > \sum_{\substack{i=M+1 \\ i \in I_+}}^{\infty} \left(\frac{1}{1+i} \right)^{2\delta} \asymp M^{-\delta}.$$

Since $\delta > 0$, this is a contradiction if M is sufficiently large. Thus, if we let N_1 be large enough so that (7.7) cannot hold whenever $M \geq N_1$, then this completes the proof of the claim. \square

Now let

$$I = I_\delta = \bigcup_{N \geq N_1-1} I_N.$$

As mentioned earlier, it is clear that I satisfies (7.2) since it satisfies (7.4). Thus to complete the proof of Theorem 7.1, it suffices to demonstrate (7.1). To this end, let $(M_k)_k$ be an increasing sequence of points at which Case 4 occurs. For each $k \in \mathbb{N}$, we have (7.6) with $M = M_k$ i.e.

$$1 - \left(\frac{2}{2+M_k} \right)^{2\delta} \leq \lambda_\delta(I_{M_k-1}) < 1.$$

Taking the limit as k approaches infinity, we see that $\lambda_\delta(I) = 1$. Thus by Bowen's formula (Theorem 3.1), we have $\text{HD}(J_I) = \delta$. This completes the proof.

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